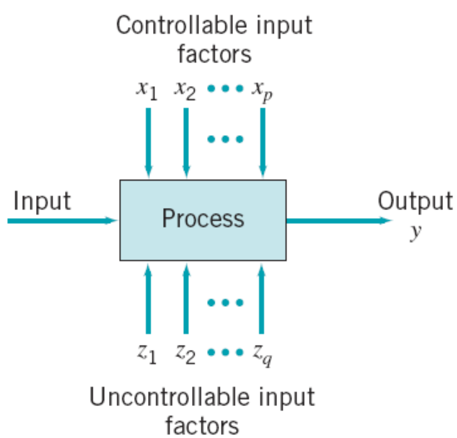
Chapter 13- Basic Experimental Design for Process Improvement

13.1 What Is Experimental Design?

As indicated in Chapter 1, a designed experiment is a test or series of tests in which purposeful changes are made to the input variables of a process so that we may observe and identify corresponding changes in the output response. The process, as shown in Fig. 13.1, can be visualized as some combination of machines, methods, and people that transforms an input material into an output product. This output product has one or more observable quality characteristics or responses. Some of the process variables $x_1, x_2, \ldots, x_p$ are **controllable,** whereas others $z_1, z_2, \ldots, z_q$ are **uncontrollable** (although they may be controllable for purposes of the test). Sometimes these uncontrollable factors are called **noise** factors. The objectives of the experiment may include

1. Determining which variables are most influential on the response, y.
2. Determining where to set the influential x's so that y is near the nominal requirement.
3. Determining where to set the influential x's so that variability in y is small.
4. Determining where to set the influential x's so that the effects of the uncontrollable variables z are minimized.



■ FIGURE 13.1 General model of a process.

EXAMPLE 13.2 Optimizing a Process

In a characterization experiment, we are usually interested in determining which process variables affect the response. A logical next step is to **optimize**—that is, to determine the region in the important factors that lead to the best possible response. For example, if the response is yield, we will look for a region of maximum yield, and if the response is variability in a critical product dimension, we will look for a region of minimum variability.

Suppose we are interested in improving the yield of a chemical process. Let's say that we know from the results of a characterization experiment that the two most important process variables that influence yield are operating temperature and reaction time. The process currently runs at 155° F and 1.7 h of reaction time, producing yields around 75%. Figure 13.2 shows a view of the time-temperature region from above. In this graph the lines of constant yield are connected to form **response contours,** and we have shown the contour lines for 60, 70, 80, 90, and 95% yield.

To locate the optimum, it is necessary to perform an experiment that varies time and temperature together. This type of experiment is called a **factorial experiment;** an example of a factorial experiment with both time and temperature run at two levels is shown in Fig. 13.2. The responses observed at the four corners of the square indicate that we should move in the general direction of increased temperature and decreased reaction time to increase yield. A few additional runs could be performed in this direction, which would be sufficient to locate the region of maximum yield. Once we are in the region of the optimum, a more elaborate experiment could be performed to give a very precise estimate of the optimum operating condition. This type of experiment, called a **response surface experiment,** is discussed in Chapter 14.

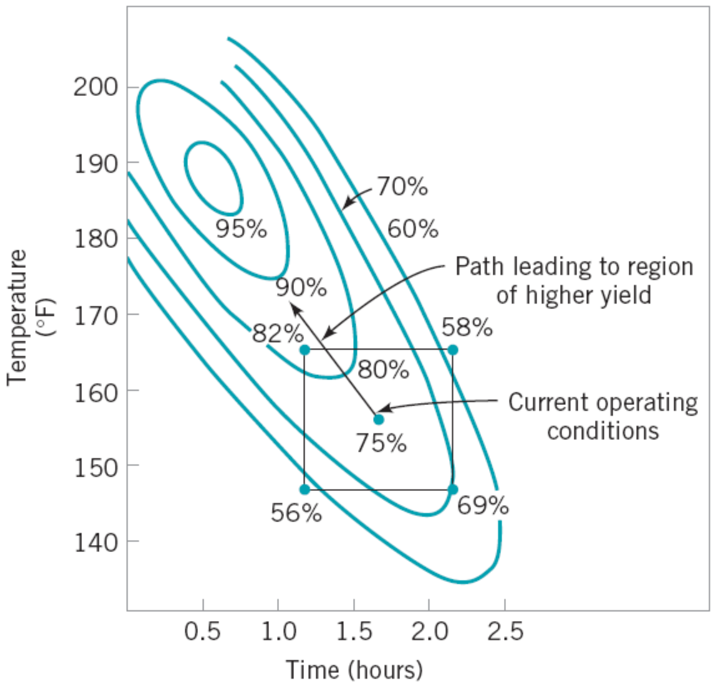


FIGURE 13.2 Contour plot of yield as a function of reaction time and reaction temperature, illustrating an optimization experiment.

Guidelines for Designing an Experiment

Pre-experimental planning

1. Recognition of and statement of the problem
2. Choice of factors and levels
3. Selection of the response variable

often done simultaneously, or in reverse order

4. Choice of experimental design
5. Performing the experiment
6. Data analysis
7. Conclusions and recommendations

■ FIGURE 13.4 Procedure for designing an experiment.

Basic Experimental Design

ANOVA

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5-8 What If We Have More Than Two Samples?

- The experimenter has a **single factor** of interest—curing methods—and there are only two **levels** of the factor.
- Many single-factor experiments require that more than two levels of the factor be considered.
- The **analysis of variance** (ANOVA) can be used for comparing means when there are more than two levels of a single factor.

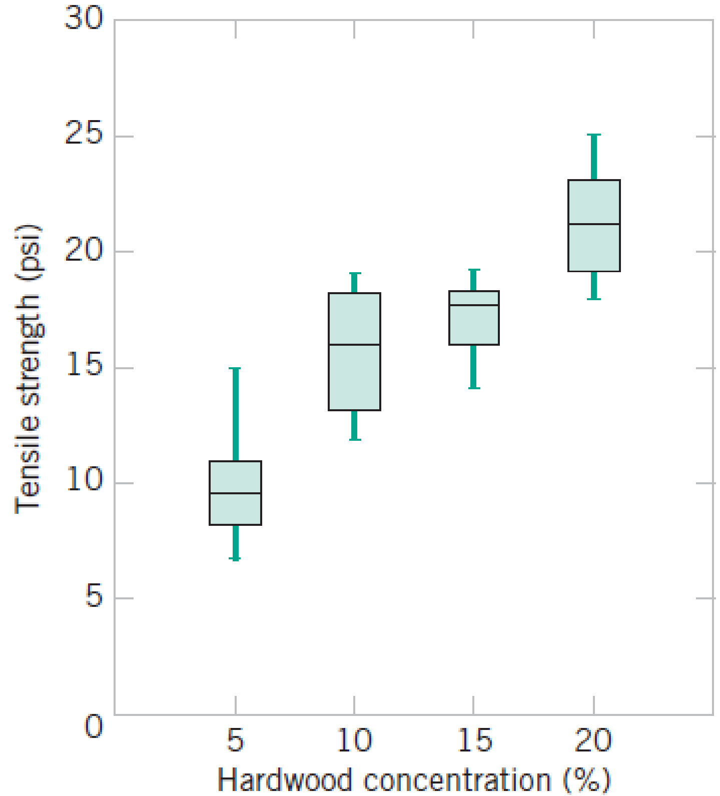
5-8 What If We Have More Than Two Samples?

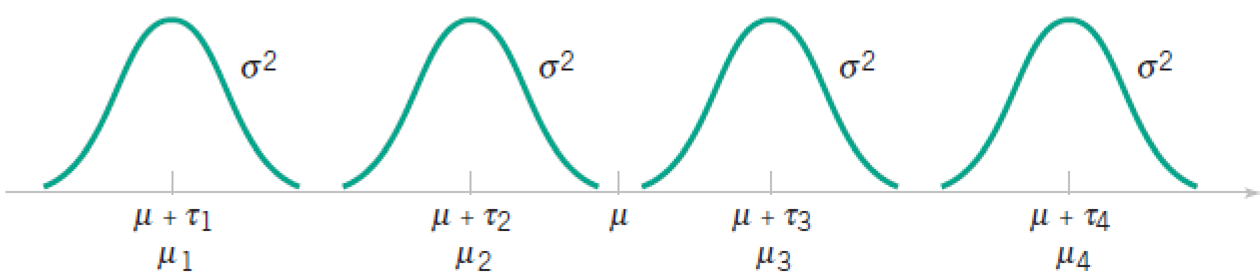
Table 5-5 Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations 1	2	3	4	5	6	Totals	Averages
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

- The levels of the factor are sometimes called **treatments**.
- Each treatment has six observations or **replicates**.
- The runs are run in **random** order.

Analysis of variance (ANOVA)





(b)

One-way ANOVA

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$$

$$H_1: \tau_i \neq 0 \quad \text{for at least one } i$$

Analysis of variance (ANOVA)

Table 5-6 Typical Data for a Single-Factor Experiment

Treatment	Observations						Totals	Averages
1	y_{11}	y_{12}	.	.	.	y_{1n}	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	y_{21}	y_{22}	.	.	.	y_{2n}	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
a	y_{a1}	y_{a2}	.	.	.	y_{an}	$y_{a\cdot}$	$\bar{y}_{a\cdot}$
							$y_{\cdot\cdot}$	$\bar{y}_{\cdot\cdot}$

y_{ij}—represents the jth observation

$$y_{i\cdot} = \sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{i\cdot} = y_{i\cdot}/n \qquad i = 1, 2, \ldots, a$$

$$y_{\cdot\cdot} = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{\cdot\cdot} = y_{\cdot\cdot}/N$$

One-way ANOVA

$$SS_T = SS_{\text{Treatments}} + SS_E \tag{5-36}$$

where

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2 = \text{total sum of squares}$$

$$SS_{\text{Treatments}} = n \sum_{i=1}^{a} (\bar{y}_{i\cdot} - \bar{y}_{..})^2 = \text{treatment sum of squares}$$

and

$$SS_E = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{i\cdot})^2 = \text{error sum of squares}$$

One-way ANOVA

Testing Hypotheses on More Than Two Means (ANOVA)

$$MS_{\text{Treatments}} = \frac{SS_{\text{Treatments}}}{a-1} \qquad E(MS_{\text{Treatments}}) = \sigma^2 + \frac{n\sum_{i=1}^{a}\tau_i^2}{a-1}$$

$$MS_E = \frac{SS_E}{a(n-1)} \qquad E(MS_E) = \sigma^2$$

Null hypothesis: $H_0\colon \tau_1 = \tau_2 = \cdots = \tau_a = 0$

Alternative hypothesis: $H_1\colon \tau_i \neq 0$ for at least one i

Test statistic: $F_0 = \dfrac{MS_{\text{Treatments}}}{MS_E}$

P-value: Probability beyond f_0 in the $F_{a-1,\,a(n-1)}$ distribution

Rejection criterion for a fixed-level test: $f_0 > f_{\alpha,a-1,a(n-1)}$ or $P\text{-}Value < \alpha$

Completely Randomized Experiment with Equal Sample Sizes

The computing formulas for the sums of squares in the analysis of variance for a completely randomized experiment with equal sample sizes in each treatment are

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N}$$

and

$$SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y_{i\cdot}^2}{n} - \frac{y_{..}^2}{N}$$

The error sum of squares is usually obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments}}$$

Table 5-7 The Analysis of Variance for a Single-Factor Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a-1$	$MS_{\text{Treatments}}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Error	SS_E	$a(n-1)$	MS_E	
Total	SS_T	$an-1$		

Example

Table 5-5 Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations						Totals	Averages
	1	2	3	4	5	6		
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

- The levels of the factor are sometimes called **treatments**.
- Each treatment has six observations or **replicates**.
- The runs are run in **random** order.

One-way ANOVA

EXAMPLE 5-14
Tensile Strength

Consider the paper tensile strength experiment described in Section 5-8.1. Use the analysis of variance to test the hypothesis that different hardwood concentrations do not affect the mean tensile strength of the paper.

$$H_0\colon \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$$

$$H_1\colon \tau_i \neq 0 \text{ for at least one } i$$

Significant level (α) =0.05

Table 5-8 Minitab Analysis of Variance Output for the Paper Tensile Strength Experiment

One-Way Analysis of Variance

$F_{\alpha,(a-1),a(n-1)}$

$F_{0.5,3,23}$

Analysis of Variance

Source	DF	SS	MS	F	P
Factor	3	382.79	127.60	19.61	0.000
Error	20	130.17	6.51		
Total	23	512.96			

$P - Value < 0.05$

Level	N	Mean	StDev
5	6	10.000	2.828
10	6	15.667	2.805
15	6	17.000	1.789
20	6	21.167	2.639

Individual 95% CIs For Mean
Based on Pooled StDev

```
----- + ---------- + ---------- + ---------- +-
 (---*---)
           (---*---)
             (---*---)
                      (---*---)
----- + ---------- + ---------- + ---------- +-
    10.0       15.0       20.0       25.0
```

Pooled StDev = 2.551

Contrasts

EXAMPLE 3-1 ....

The Tensile Strength Experiment

To illustrate the analysis of variance, return to the example first discussed in Section 3-1. Recall that the development engineer is interested in determining if the cotton weight percentage in a synthetic fiber affects the tensile strength, and she has run a completely randomized experiment with five levels of cotton weight percentage and five replicates. For convenience, we repeat the data from Table 3-1 here:

Weight Percentage of Cotton	Observed Tensile Strength (lb/in^2)					Totals	Averages
	1	2	3	4	5	$y_{i.}$	$\bar{y}_{i.}$
15	7	7	15	11	9	49	9.8
20	12	17	12	18	18	77	15.4
25	14	18	18	19	19	88	17.6
30	19	25	22	19	23	108	21.6
35	7	10	11	15	11	54	10.8
						$y_{..} = 376$	$\bar{y}_{..} = 15.04$

$$SS_T = \sum_{i=1}^{5} \sum_{j=1}^{5} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$= (7)^2 + (7)^2 + (15)^2 + \cdots + (15)^2 + (11)^2 - \frac{(376)^2}{25} = 636.96$$

$$SS_{\text{Treatments}} = \frac{1}{n} \sum_{i=1}^{5} y_{i.}^2 - \frac{y_{..}^2}{N}$$

$$= \frac{1}{5}[(49)^2 + \cdots + (54)^2] - \frac{(376)^2}{25} = 475.76$$

$$SS_E = SS_T - SS_{\text{Treatments}}$$

$$= 636.96 - 475.76 = 161.20$$

Table 3-4 Analysis of Variance for the Tensile Strength Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Cotton weight percentage	475.76	4	118.94	$F_0 = 14.76$	<0.01
Error	161.20	20	8.06		
Total	636.96	24			

Contrasts

Many multiple comparison methods use the idea of a contrast. Consider the synthetic fiber testing problem of Example 3-1. Because the null hypothesis was rejected, we know that some cotton weight percentages produce different tensile strengths than others, but which ones actually cause this difference? We might suspect at the outset of the experiment that levels 4 and 5 cotton weight percentages (30 and 35 percent) produce the same tensile strength, implying that we would like to test the hypothesis

$$\begin{aligned} H_0&: \mu_4 = \mu_5 \\ H_1&: \mu_4 \neq \mu_5 \end{aligned}$$

or equivalently,

$$\begin{aligned} H_0&: \mu_4 - \mu_5 = 0 \\ H_1&: \mu_4 - \mu_5 \neq 0 \end{aligned} \tag{3-23}$$

If we had suspected at the start of the experiment that the *average* of the lowest levels of cotton weight percentages (1 and 2) did not differ from the *average* of the highest levels of cotton weight percentages (4 and 5), then the hypothesis would have been

$$H_0: \mu_1 + \mu_2 = \mu_4 + \mu_5$$
$$H_1: \mu_1 + \mu_2 \neq \mu_4 + \mu_5$$

or

$$H_0: \mu_1 + \mu_2 - \mu_4 - \mu_5 = 0 \qquad (3\text{-}24)$$
$$H_1: \mu_1 + \mu_2 - \mu_4 - \mu_5 \neq 0$$

In general, a **contrast** is a linear combination of parameters of the form

$$\Gamma = \sum_{i=1}^{a} c_i \mu_i$$

where the **contrast constant** $c_1, c_2, \ldots, c_a$ sum to zero; that is, $\Sigma_{i=1}^{a} c_i = 0$. Both of the hypotheses above can be expressed in terms of contrasts:

$$H_0: \sum_{i=1}^{a} c_i \mu_i = 0 \qquad (3\text{-}25)$$
$$H_1: \sum_{i=1}^{a} c_i \mu_i \neq 0$$

Example

The contrast constants for the hypotheses in Equation 3-23 are $c_1 = c_2 = c_3 = 0$, $c_4 = +1$, and $c_5 = -1$, whereas for the hypotheses in Equation 3-24 they are $c_1 = c_2 = +1$, $c_3 = 0$, and $c_4 = c_5 = -1$.

Testing hypotheses involving contrasts can be done in two basic ways. The first method uses a t-test. Write the contrast of interest in terms of the **treatment totals**, giving

$$C = \sum_{i=1}^{a} c_i y_{i.}$$

The second approach uses an F test. Now the square of a t random variable with ν degrees of freedom is an F random variable with 1 numerator and ν denominator degrees of freedom. Therefore, we can obtain

$$F_0 = t_0^2 = \frac{\left(\sum_{i=1}^{a} c_i y_{i.}\right)^2}{nMS_E \sum_{i=1}^{a} c_i^2} \tag{3-28}$$

as an F statistic for testing Equation 3-25. The null hypothesis would be rejected if $F_0 > F_{\alpha,1,N-a}$. We can write the test statistic of Equation 3-28 as

$$F_0 = \frac{MS_C}{MS_E} = \frac{SS_C/1}{MS_E}$$

where the single degree of freedom contrast sum of squares is

$$SS_C = \frac{\left(\sum_{i=1}^{a} c_i y_{i.}\right)^2}{n \sum_{i=1}^{a} c_i^2} \tag{3-29}$$

Example

EXAMPLE 3-6

Consider the data in Example 3-1. There are five treatment means and four degrees of freedom between these treatments. Suppose that prior to running the experiment the following set of comparisons among the treatment means (and their associated contrasts) were specified:

Hypothesis	Contrast
$H_0: \mu_4 = \mu_5$	$C_1 = -y_{4.} + y_{5.}$
$H_0: \mu_1 + \mu_3 = \mu_4 + \mu_5$	$C_2 = y_{1.} + y_{3.} - y_{4.} - y_{5.}$
$H_0: \mu_1 = \mu_3$	$C_3 = y_{1.} - y_{3.}$
$H_0: 4\mu_2 = \mu_1 + \mu_3 + \mu_4 + \mu_5$	$C_4 = -y_{1.} + 4y_{2.} - y_{3.} - y_{4.} - y_{5.}$

Notice that the contrast coefficients are orthogonal. Using the data in Table 3-4, we find the numerical values of the contrasts and the sums of squares to be as follows:

$$C_1 = \qquad - 1(108) + 1(54) = -54 \quad SS_{C_1} = \frac{(-54)^2}{5(2)} = 291.60$$

$$C_2 = +1(49) \qquad + 1(88) - 1(108) - 1(54) = -25 \quad SS_{C_2} = \frac{(-25)^2}{5(4)} = 31.25$$

$$C_3 = +1(49) \qquad - 1(88) \qquad = -39 \quad SS_{C_3} = \frac{(-39)^2}{5(2)} = 152.10$$

$$C_4 = -1(49) + 4(77) - 1(88) - 1(108) - 1(54) = 9 \quad SS_{C_4} = \frac{(9)^2}{5(20)} = 0.81$$

Table 3-11 Analysis of Variance for the Tensile Strength Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Cotton Weight Percentage	475.76	4	118.94	14.76	<0.001
orthogonal contrasts					
$C_1: \mu_4 = \mu_5$	(291.60)	1	291.60	36.18	<0.001
$C_2: \mu_1 + \mu_3 = \mu_4 + \mu_5$	(31.25)	1	31.25	3.88	0.06
$C_3: \mu_1 = \mu_3$	(152.10)	1	152.10	18.87	<0.001
$C_4: 4\mu_2 = \mu_1 + \mu_3 + \mu_4 + \mu_5$	(0.81)	1	0.81	0.10	0.76
Error	161.20	20	8.06		
Total	636.96	24			

Exercise 1

- ในการศึกษาว่าร้านชุบเหล็กด้วยสังกะสี 3 แห่ง ทำการชุบเหล็กด้วยสังกะสีให้ค่าความหนาของการชุบ (หน่วยเป็น mm) แตกต่างกันหรือไม่ ทดสอบที่ระดับนัยสำคัญ 5%

ร้าน				
A	40	38	30	47
B	25	32	13	35
C	27	24	20	13

Exercise 2

- ในการศึกษาอายุการใช้งานของแบตเตอรี่ 3 ยี่ห้อ แต่ละยี่ห้อใช้แบตเตอรี่ 3 อันได้ผลของอายุการใช้งาน (สัปดาห์) ดังตารางต่อไปนี้

ก. อายุการใช้งานของแบตเตอรี่ทั้ง 3 ยี่ห้อนี้ แตกต่างกันหรือไม่

ข. ท่านควรเลือกใช้แบตเตอรี่ยี่ห้อใดและหากทางโรงงานสัญญาว่าจะเปลี่ยนแบตเตอรี่อันใหม่ให้หากใช้ได้ไม่ถึง 80 สัปดาห์ จงหาร้อยละของจำนวนแบตเตอรี่ที่คาดว่าจะต้องเปลี่ยนใหม่

แบตเตอรี่ 1	100	96	92
แบตเตอรี่ 2	76	80	75
แบตเตอรี่ 3	108	100	96

Minitab result

One-way ANOVA: A, B, C

Source	DF	SS	MS	F	P
Factor	2	665.2	332.6	5.51	0.027
Error	9	543.5	60.4		
Total	11	1208.7			

S = 7.771   R-Sq = 55.03%   R-Sq(adj) = 45.04%

```
                         Individual 95% CIs For Mean Based on
                         Pooled StDev
Level  N   Mean   StDev  --------+---------+---------+---------+-
  A    4  38.750   6.994                  (--------*--------)
  B    4  26.250  9.777         (--------*--------)
  C    4  21.000  6.055  (--------*--------)
                         --------+---------+---------+---------+-
                                20        30        40        50
```

Pooled StDev = 7.771

One-way ANOVA: Batery 1, Battery2, Battery3

Source	DF	SS	MS	F	P
Factor	2	981.6	490.8	24.40	0.001
Error	6	120.7	20.1		
Total	8	1102.2			

S = 4.485   R-Sq = 89.05%   R-Sq(adj) = 85.40%

```
                                 Individual 95% CIs For Mean Based on
                                 Pooled StDev
Level      N    Mean  StDev  ---------+---------+---------+---------+
Batery 1   3   96.00   4.00                   (-----*-----)
Battery2   3   77.00   2.65  (-----*-----)
Battery3   3  101.33   6.11                        (-----*------)
                             ---------+---------+---------+---------+
                                     80        90       100       110
```

Pooled StDev = 4.48

- $P(Y_2<80) = P(Z<(80-77)/2.65)$

  $= P(Z<1.132)$

  $= 0.871$

  $= 87.1\%$

The Fisher Least Significant Difference (LSD) Method

This procedure uses the F statistic for testing $H_0: \mu_i = \mu_j$

$$t_0 = \frac{\bar{y}_{i.} - \bar{y}_{j.}}{\sqrt{MS_E\left(\dfrac{1}{n_i} + \dfrac{1}{n_j}\right)}} \tag{3-39}$$

Assuming a two-sided alternative, the pair of means μ_i and μ_j would be declared significantly different if $|\bar{y}_{i.} - \bar{y}_{j.}| > t_{\alpha/2,N-a}\sqrt{MS_E(1/n_i + 1/n_j)}$. The quantity

$$\text{LSD} = t_{\alpha/2,N-a}\sqrt{MS_E\left(\frac{1}{n_i} + \frac{1}{n_j}\right)} \tag{3-40}$$

is called the **least significant difference**. If the design is balanced, $n_1 = n_2 = \cdots = n_a = n$, and

$$\text{LSD} = t_{\alpha/2,N-a}\sqrt{\frac{2MS_E}{n}} \tag{3-41}$$

To use the Fisher LSD procedure, we simply compare the observed difference between each pair of averages to the corresponding LSD. If $|\bar{y}_{i.} - \bar{y}_{j.}| > \text{LSD}$, we conclude that the population means μ_i and μ_j differ.

EXAMPLE 3-8 ....

To illustrate the procedure, if we use the data from the experiment in Example 3-1, the LSD at $\alpha = 0.05$ is

$$\text{LSD} = t_{.025,20}\sqrt{\frac{2MS_E}{n}} = 2.086\sqrt{\frac{2(8.06)}{5}} = 3.75$$

Thus, any pair of treatment averages that differ in absolute value by more than 3.75 would imply that the corresponding pair of population means are significantly different. The differences in averages are

$$\begin{aligned}
\bar{y}_{1.} - \bar{y}_{2.} &= 9.8 - 15.4 = -5.6^* \\
\bar{y}_{1.} - \bar{y}_{3.} &= 9.8 - 17.6 = -7.8^* \\
\bar{y}_{1.} - \bar{y}_{4.} &= 9.8 - 21.6 = -11.8^* \\
\bar{y}_{1.} - \bar{y}_{5.} &= 9.8 - 10.8 = -1.0 \\
\bar{y}_{2.} - \bar{y}_{3.} &= 15.4 - 17.6 = -2.2 \\
\bar{y}_{2.} - \bar{y}_{4.} &= 15.4 - 21.6 = -6.2^* \\
\bar{y}_{2.} - \bar{y}_{5.} &= 15.4 - 10.8 = 4.6^* \\
\bar{y}_{3.} - \bar{y}_{4.} &= 17.6 - 21.6 = -4.0^* \\
\bar{y}_{3.} - \bar{y}_{5.} &= 17.6 - 10.8 = 6.8^* \\
\bar{y}_{4.} - \bar{y}_{5.} &= 21.6 - 10.8 = 10.8^*
\end{aligned}$$

$\bar{y}_{1.}$	$\bar{y}_{5.}$	$\bar{y}_{2.}$	$\bar{y}_{3.}$	$\bar{y}_{4.}$
9.8	10.8	15.4	17.6	21.6

Figure 3-13 Results of the LSD procedure.

The starred values indicate pairs of means that are significantly different. Figure 3-13 summarizes the results. Clearly, the only pairs of means that do not differ significantly are 1 and 5 and 2 and 3, and treatment 4 produces a significantly greater tensile strength than the other treatments.

Duncan's Multiple Range Test

A widely used procedure for comparing all pairs of means is the **multiple range test** developed by Duncan (1955). To apply Duncan's multiple range test for equal sample sizes, the a treatment averages are arranged in ascending order, and the standard error of each average is determined as

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{n}} \tag{3-42}$$

For unequal sample sizes, replace n in Equation 3-42 by the harmonic mean n_h of the $\{n_i\}$, where

$$n_h = \frac{a}{\sum_{i=1}^{a} (1/n_i)} \tag{3-43}$$

Note that if $n_1 = n_2 = \cdots = n_a$, $n_h = n$. From Duncan's table of significant ranges (Appendix Table VII), obtain the values $r_\alpha(p, f)$ for $p = 2, 3, \ldots, a$, where α is the significance level and f is the number of degrees of freedom for error. Convert these ranges into a set of $a - 1$ least significant ranges (e.g., R_p) for $p = 2, 3, \ldots, a$ by calculating

$$R_p = r_\alpha(p, f) S_{\bar{y}_{i.}} \qquad \text{for } p = 2, 3, \ldots, a \tag{3-44}$$

Example

EXAMPLE 3-9 ....

We can apply Duncan's multiple range test to the experiment of Example 3-1. Recall that $MS_E = 8.06$, $N = 25$, $n = 5$, and there are 20 error degrees of freedom. Ranking the treatment averages in ascending order, we have

$$\begin{aligned}
\bar{y}_{1.} &= 9.8 \\
\bar{y}_{5.} &= 10.8 \\
\bar{y}_{2.} &= 15.4 \\
\bar{y}_{3.} &= 17.6 \\
\bar{y}_{4.} &= 21.6
\end{aligned}$$

The standard error of each average is $S_{\bar{y}_{i.}} = \sqrt{8.06/5} = 1.27$. From the table of significant ranges in Appendix Table VII for 20 degrees of freedom and $\alpha = 0.05$, we obtain $r_{0.05}(2, 20) = 2.95$, $r_{0.05}(3, 20) = 3.10$, $r_{0.05}(4, 20) = 3.18$, and $r_{0.05}(5, 20) = 3.25$. Thus, the least significant ranges are

$$
\begin{aligned}
R_2 &= r_{0.05}(2, 20)S_{\bar{y}_{i.}} = (2.95)(1.27) = 3.75 \\
R_3 &= r_{0.05}(3, 20)S_{\bar{y}_{i.}} = (3.10)(1.27) = 3.94 \\
R_4 &= r_{0.05}(4, 20)S_{\bar{y}_{i.}} = (3.18)(1.27) = 4.04 \\
R_5 &= r_{0.05}(5, 20)S_{\bar{y}_{i.}} = (3.25)(1.27) = 4.13
\end{aligned}
$$

The comparisons would yield

$$
\begin{aligned}
&4 \text{ vs. } 1\text{: } 21.6 - 9.8 = 11.8 > 4.13(R_5) \\
&4 \text{ vs. } 5\text{: } 21.6 - 10.8 = 10.8 > 4.04(R_4) \\
&4 \text{ vs. } 2\text{: } 21.6 - 15.4 = 6.2 > 3.94(R_3) \\
&4 \text{ vs. } 3\text{: } 21.6 - 17.6 = 4.0 > 3.75(R_2) \\
&3 \text{ vs. } 1\text{: } 17.6 - 9.8 = 7.8 > 4.04(R_4) \\
&3 \text{ vs. } 5\text{: } 17.6 - 10.8 = 6.8 > 3.95(R_3) \\
&3 \text{ vs. } 2\text{: } 17.6 - 15.4 = 2.2 < 3.75(R_2) \\
&2 \text{ vs. } 1\text{: } 15.4 - 9.8 = 5.6 > 3.94(R_3) \\
&2 \text{ vs. } 5\text{: } 15.4 - 10.8 = 4.6 > 3.75(R_2) \\
&5 \text{ vs. } 1\text{: } 10.8 - 9.8 = 1.0 < 3.75(R_2)
\end{aligned}
$$

From the analysis we see that there are significant differences between all pairs of means except 3 and 2 and 5 and 1. A graph underlying those means that are not significantly different is shown in Figure 3-14. Notice that, in this example, Duncan's multiple range test and the LSD method produce identical conclusions.

$\bar{y}_{1.}$	$\bar{y}_{5.}$	$\bar{y}_{2.}$	$\bar{y}_{3.}$	$\bar{y}_{4.}$
9.8	10.8	15.4	17.6	21.6

Figure 3-14 Results of Duncan's multiple range test.

$r_{0.01}(p, f)$

f	p = 2	3	4	5	6	7	8	9	10	20	50	100
1	90.0	90.0	90.0	90.0	90.0	90.0	90.0	90.0	90.0	90.0	90.0	90.0
2	14.0	14.0	14.0	14.0	14.0	14.0	14.0	14.0	14.0	14.0	14.0	14.0
3	8.26	8.5	8.6	8.7	8.8	8.9	8.9	9.0	9.0	9.3	9.3	9.3
4	6.51	6.8	6.9	7.0	7.1	7.1	7.2	7.2	7.3	7.5	7.5	7.5
5	5.70	5.96	6.11	6.18	6.26	6.33	6.40	6.44	6.5	6.8	6.8	6.8
6	5.24	5.51	5.65	5.73	5.81	5.88	5.95	6.00	6.0	6.3	6.3	6.3
7	4.95	5.22	5.37	5.45	5.53	5.61	5.69	5.73	5.8	6.0	6.0	6.0
8	4.74	5.00	5.14	5.23	5.32	5.40	5.47	5.51	5.5	5.8	5.8	5.8
9	4.60	4.86	4.99	5.08	5.17	5.25	5.32	5.36	5.4	5.7	5.7	5.7
10	4.48	4.73	4.88	4.96	5.06	5.13	5.20	5.24	5.28	5.55	5.55	5.55
11	4.39	4.63	4.77	4.86	4.94	5.01	5.06	5.12	5.15	5.39	5.39	5.39
12	4.32	4.55	4.68	4.76	4.84	4.92	4.96	5.02	5.07	5.26	5.26	5.26
13	4.26	4.48	4.62	4.69	4.74	4.84	4.88	4.94	4.98	5.15	5.15	5.15
14	4.21	4.42	4.55	4.63	4.70	4.78	4.83	4.87	4.91	5.07	5.07	5.07
15	4.17	4.37	4.50	4.58	4.64	4.72	4.77	4.81	4.84	5.00	5.00	5.00
16	4.13	4.34	4.45	4.54	4.60	4.67	4.72	4.76	4.79	4.94	4.94	4.94
17	4.10	4.30	4.41	4.50	4.56	4.63	4.68	4.73	4.75	4.89	4.89	4.89
18	4.07	4.27	4.38	4.46	4.53	4.59	4.64	4.68	4.71	4.85	4.85	4.85
19	4.05	4.24	4.35	4.43	4.50	4.56	4.61	4.64	4.67	4.82	4.82	4.82
20	4.02	4.22	4.33	4.40	4.47	4.53	4.58	4.61	4.65	4.79	4.79	4.79
30	3.89	4.06	4.16	4.22	4.32	4.36	4.41	4.45	4.48	4.65	4.71	4.71
40	3.82	3.99	4.10	4.17	4.24	4.30	4.34	4.37	4.41	4.59	4.69	4.69
60	3.76	3.92	4.03	4.12	4.17	4.23	4.27	4.31	4.34	4.53	4.66	4.66
100	3.71	3.86	3.98	4.06	4.11	4.17	4.21	4.25	4.29	4.48	4.64	4.65
∞	3.64	3.80	3.90	3.98	4.04	4.09	4.14	4.17	4.20	4.41	4.60	4.68

f = degrees of freedom.

[a] Reproduced with permission from "Multiple Range and Multiple F Tests," by D. B. Duncan, *Biometrics*, Vol. 1, No. 1, pp. 1–42, 1955.

$r_{0.05}(p, f)$

f	p = 2	3	4	5	6	7	8	9	10	20	50	100
1	18.0	18.0	18.0	18.0	18.0	18.0	18.0	18.0	18.0	18.0	18.0	18.0
2	6.09	6.09	6.09	6.09	6.09	6.09	6.09	6.09	6.09	6.09	6.09	6.09
3	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50
4	3.93	4.01	4.02	4.02	4.02	4.02	4.02	4.02	4.02	4.02	4.02	4.02
5	3.64	3.74	3.79	3.83	3.83	3.83	3.83	3.83	3.83	3.83	3.83	3.83
6	3.46	3.58	3.64	3.68	3.68	3.68	3.68	3.68	3.68	3.68	3.68	3.68
7	3.35	3.47	3.54	3.58	3.60	3.61	3.61	3.61	3.61	3.61	3.61	3.61
8	3.26	3.39	3.47	3.52	3.55	3.56	3.56	3.56	3.56	3.56	3.56	3.56
9	3.20	3.34	3.41	3.47	3.50	3.52	3.52	3.52	3.52	3.52	3.52	3.52
10	3.15	3.30	3.37	3.43	3.46	3.47	3.47	3.47	3.47	3.48	3.48	3.48
11	3.11	3.27	3.35	3.39	3.43	3.44	3.45	3.46	3.46	3.48	3.48	3.48
12	3.08	3.23	3.33	3.36	3.40	3.42	3.44	3.44	3.46	3.48	3.48	3.48
13	3.06	3.21	3.30	3.35	3.38	3.41	3.42	3.44	3.45	3.47	3.47	3.47
14	3.03	3.18	3.27	3.33	3.37	3.39	3.41	3.42	3.44	3.47	3.47	3.47
15	3.01	3.16	3.25	3.31	3.36	3.38	3.40	3.42	3.43	3.47	3.47	3.47
16	3.00	3.15	3.23	3.30	3.34	3.37	3.39	3.41	3.43	3.47	3.47	3.47
17	2.98	3.13	3.22	3.28	3.33	3.36	3.38	3.40	3.42	3.47	3.47	3.47
18	2.97	3.12	3.21	3.27	3.32	3.35	3.37	3.39	3.41	3.47	3.47	3.47
19	2.96	3.11	3.19	3.26	3.31	3.35	3.37	3.39	3.41	3.47	3.47	3.47
20	2.95	3.10	3.18	3.25	3.30	3.34	3.36	3.38	3.40	3.47	3.47	3.47
30	2.89	3.04	3.12	3.20	3.25	3.29	3.32	3.35	3.37	3.47	3.47	3.47
40	2.86	3.01	3.10	3.17	3.22	3.27	3.30	3.33	3.35	3.47	3.47	3.47
60	2.83	2.98	3.08	3.14	3.20	3.24	3.28	3.31	3.33	3.47	3.48	3.48
100	2.80	2.95	3.05	3.12	3.18	3.22	3.26	3.29	3.32	3.47	3.53	3.53
∞	2.77	2.92	3.02	3.09	3.15	3.19	3.23	3.26	3.29	3.47	3.61	3.67

Exercise

โรงทอผ้ามีเครื่องทอผ้า (Machine) จำนวน หลายเครื่อง แต่ละเครื่องควรจะมี ประสิทธิภาพการทำงาน (Performance) ซึ่งวัดจากจำนวนผ้าที่ผลิตได้ต่อนาที ทีเท่าๆ กัน เพื่อที่จะทดสอบสมมติฐานนี้ เครื่องทอผ้า 5 เครื่องได้เลือกมาเพื่อวัดค่าประสิทธภาพการทำงานได้ผลดังตารางด้านล่าง

Machine	Performance (lb/in^2)				
	1	2	3	4	5
1	14	14.1	14.2	14	14.1
2	13.9	13.8	13.9	14	14
3	14.1	14.2	14.1	14	13.9
4	13.6	13.8	14	13.9	13.7
5	13.8	13.6	13.9	13.8	14

ผลการวิเคราะห์ความแปรปรวนแสดงดังต่อไปนี้

One-way ANOVA: Performance versus Machine

```
Analysis of Variance for Performa
Source      DF        SS        MS        F        P
Machine      4    0.3416    0.0854     5.77    0.003
Error       20    0.2960    0.0148
Total       24    0.6376
                                   Individual 95% CIs For Mean
                                   Based on Pooled StDev
Level        N      Mean     StDev  --------+---------+---------+--------
1            5    14.080     0.084                   (-------*------)
2            5    13.920     0.084          (-------*-------)
3            5    14.060     0.114                  (------*-------)
4            5    13.800     0.158  (-------*-------)
5            5    13.820     0.148    (------*-------)
                                    --------+---------+---------+--------
Pooled StDev =    0.122                   13.80     13.95     14.10

Fisher's pairwise comparisons
```

จงสรุปผลการทดลองนี้ ด้วยเลขนัยสำคัญ (α) = 0.10 และจากการทดลองจงใช้การทดสอบวิธี Least Significant Different ที่ระดับ $\alpha = 0.05$ เพื่อเปรียบเทียบระหว่างค่าเฉลี่ยแต่ละค่า และสรุปผลเลือกเครื่องจักรที่ให้ประสิทธิภาพสูงสุด หมายเหตุ: ถ้าในตารางไม่มีค่าที่ต้องการให้เลือกค่าที่ใกล้เคียง